ON $(h-m)-$CONVEXITY AND HADAMARD-TYPE INEQUALITIES

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Abstract. In this paper, a new class of convex functions as a generalization of convexity which is called $(h-m)-$convex functions and some properties of this class is given. We also prove some Hadamard's type inequalities.

1. INTRODUCTION

The concept of $m-$convexity has been introduced by Toader in [12], an intermediate between the ordinary convexity and starshaped property, as following:

Definition 1. *The function $f:[0,b]\to\mathbb{R}$, $b>0$, is said to be $m-$convex, where $m\in[0,1]$, if we have*

$$f(tx+m(1-t)y)\leq tf(x)+m(1-t)f(y)$$

for all $x,y\in[0,b]$ and $t\in[0,1]$. We say that f is $m-$concave if $-f$ is $m-$convex.

Several papers have been written on $m-$convex functions and we refer the papers [11], [12], [13], [14], [15] and [17]. In [13], Dragomir and Toader proved following inequality for $m-$convex functions.

Theorem 1. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)\,dx\leq\min\left\{\frac{f(a)+mf\left(\frac{b}{m}\right)}{2},\frac{f(b)+mf\left(\frac{a}{m}\right)}{2}\right\}. \tag{1.1}$$

In [17], Dragomir established following inequalities of Hadamard-type similar to above.

Theorem 2. *Let $f:[0,\infty)\to\mathbb{R}$ be a $m-$convex function with $m\in(0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]$, then one has the inequality:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b\frac{f(x)+mf\left(\frac{x}{m}\right)}{2}dx \\ &\leq \frac{m+1}{4}\left[\frac{f(a)+f(b)}{2}+m\frac{f(\frac{a}{m})+f\left(\frac{b}{m}\right)}{2}\right]. \end{aligned} \tag{1.2}$$

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Theorem 3. *Let $f : [0,\infty) \to \mathbb{R}$ be a $m-$convex function with $m \in (0,1]$. If $f \in L_1[am,b]$ where $0 \leq a < b < \infty$, then one has the inequality:*

$$\frac{1}{m+1}\left[\int_a^{mb} f(x)dx + \frac{mb-a}{b-ma}\int_{ma}^{b} f(x)dx\right] \leq (mb-a)\,\frac{f(a)+f(b)}{2}. \tag{1.3}$$

In [16], Breckner introduced a new class of convex functions, a generalization of the ordinary convexity, is called $s-$convexity, as following;

Definition 2. *Let s be a real number, $s \in (0,1]$. A function $f : [0,\infty) \to [0,\infty)$ is said to be $s-$convex (in the second sense), or that f belongs to the class K_s^2, if*

$$f(\alpha x + (1-\alpha)y) \leq \alpha^s f(x) + (1-\alpha)^s f(y)$$

for all $x, y \in [0,\infty)$ and $\alpha \in [0,1]$.

Some properties of $s-$convexity have been given in [4] and Kırmacı *et al.* proved some inequalities for $s-$convex functions in [10]. In [9], Dragomir and Fitzpatrick established the following Hadamard's type inequalities;

Theorem 4. *Suppose that $f : [0,\infty) \to [0,\infty)$ is an $s-$convex function in the second sense, where $s \in (0,1]$ and let $a, b \in [0,\infty)$, $a < b$. If $f \in L_1[0,1]$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.4}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.4). The above inequalities are sharp.

In [3], Godunova and Levin introduced the following class of functions.

Definition 3. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to belong to the class of $Q(I)$ if it is nonnegative and, for all $x, y \in I$ and $\lambda \in (0,1)$ satisfies the inequality;*

$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}$$

In [2], Dragomir *et al.* defined following new class of functions.

Definition 4. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is P function or that f belongs to the class of $P(I)$, if it is nonnegative and for all $x, y \in I$ and $\lambda \in [0,1]$, satisfies the following inequality;*

$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y)$$

In [2], Dragomir *et al.* proved two inequalities of Hadamard's type for class of Godunova-Levin functions and $P-$ functions.

Theorem 5. *Let $f \in Q(I), a, b \in I$, with $a < b$ and $f \in L_1[a,b]$. Then the following inequality holds.*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a}\int_a^b f(x)dx \tag{1.5}$$

Theorem 6. *Let $f \in P(I), a, b \in I$, with $a < b$ and $f \in L_1[a,b]$. Then the following inequality holds.*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\int_a^b f(x)dx \leq 2[f(a)+f(b)] \tag{1.6}$$

On all of these, in [5], Varošanec defined $h-$convex functions and gave some properties of this class of functions.

Definition 5. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a positive function . We say that $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is $h-$convex function, or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx + (1-t)y) \leq h(t) f(x) + h(1-t) f(y). \tag{1.7}$$

If inequality (1.7) is reversed, then f is said to be $h-$concave, i.e., $f \in SV(h, I)$. Obviously, if $h(t) = t$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(t) = \frac{1}{t}$, then $SX(h, I) = Q(I)$; if $h(t) = 1$, $SX(h, I) \supseteq P(I)$; and if $h(t) = t^s$, where $s \in (0,1)$, then $SX(h, I) \supseteq K_s^2$.

Theorem 7. *(See [1], Theorem 6) Let $f \in SX(h, I)$, $a, b \in I$, with $a < b$ and $f \in L_1([a,b])$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)}\int_a^b f(x)\,dx \leq [f(a)+f(b)]\int_0^1 h(\alpha)\,d\alpha \tag{1.8}$$

For some recent results for $h-$convex functions we refer the interest of reader to the papers [1], [6], [7] and [8].

The main aim of this paper is to give a new class of convex functions and to give some properties of this functions, by using a similar way to proof of properties of $h-$convexity (see [5]). Therefore, some inequalities of Hadamard-type related to this new class of convex functions are given.

2. MAIN RESULTS

We will introduce a new class of convex functions in the following definition.

Definition 6. *Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a non-negative function. We say that $f : [0,b] \to \mathbb{R}$ is a $(h-m)-$convex function , if f is non-negative and for all $x, y \in [0,b]$, $m \in [0,1]$ and $\alpha \in (0,1)$, we have*

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha) f(x) + m h(1-\alpha) f(y). \tag{2.1}$$

If the inequality (2.1) is reversed, then f is said to be $(h-m)-$concave function on $[0,b]$.

Obviously, if we choose $m = 1$, then we have $h-$convex functions. If we choose $h(\alpha) = \alpha$, then we obtain non-negative $m-$convex functions. If we choose $m = 1$ and $h(\alpha) = \left\{\alpha,\ 1,\ \frac{1}{\alpha},\ \alpha^s\right\}$, then we obtain the following classes of functions, non-negative convex functions, $P-$functions, Godunova-Levin functions and $s-$convex functions (in the second sense), respectively.

Remark 1. *Let h be a non-negative function such that*

$$h(\alpha) \geq \alpha$$

for all $\alpha \in (0,1)$*. If* f *is a non-negative* $m-$*convex function on* $[0,b]$*, then for all* $x,y \in [0,b]$*,* $m \in [0,1]$ *and* $\alpha \in (0,1)$*, we have*

$$f(\alpha x + m(1-\alpha)y) \leq \alpha f(x) + m(1-\alpha)f(y) \leq h(\alpha)f(x) + mh(1-\alpha)f(y).$$

This shows that f *is a* $(h-m)-$*convex function. By a similar way, one can see that, if*

$$h(\alpha) \leq \alpha$$

for all $\alpha \in (0,1)$*. Then, all non-negative* $m-$*concave functions are* $(h-m)-$*concave function on* $[0,b]$*.*

Proposition 1. *Let* h_1, h_2 *be non-negative functions defined on* $J \subseteq \mathbb{R}$ *such that*

$$h_2(t) \leq h_1(t)$$

for $t \in (0,1)$*. If* f *is* $(h_2-m)-$*convex, then* f *is* $(h_1-m)-$*convex.*

Proof. If f is $(h_2-m)-$convex, then for all $x,y \in [0,b]$ and $\alpha \in (0,1)$, we can write

$$\begin{aligned} f(\alpha x + m(1-\alpha)y) &\leq h_2(\alpha)f(x) + mh_2(1-\alpha)f(y) \\ &\leq h_1(\alpha)f(x) + mh_1(1-\alpha)f(y). \end{aligned}$$

Which completes the proof of $(h_1-m)-$convexity of f. □

Proposition 2. *If* f,g *are* $(h-m)-$*convex functions and* $\lambda > 0$*, then* $f+g$ *and* λf *are* $(h-m)-$*convex functions.*

Proof. By using definition of $(h-m)-$convex functions, we can write

$$f(\alpha x + m(1-\alpha)y) \leq h(\alpha)f(x) + mh(1-\alpha)f(y) \tag{2.2}$$

and

$$g(\alpha x + m(1-\alpha)y) \leq h(\alpha)g(x) + mh(1-\alpha)g(y) \tag{2.3}$$

for all $x,y \in [0,b]$, $m \in [0,1]$ and $\alpha \in (0,1)$. If we add (2.2) and (2.3), we get

$$(f+g)(\alpha x + m(1-\alpha)y) \leq h(\alpha)(f+g)(x) + mh(1-\alpha)(f+g)(y).$$

This shows that $f+g$ is $(h-m)-$convex function. Therefore, to prove $(h-m)-$convexity of λf, from the definition, we have

$$\lambda f(\alpha x + m(1-\alpha)y) \leq h(\alpha)\lambda f(x) + mh(1-\alpha)\lambda f(y).$$

This completes the proof. □

The following inequality of Hadamard-type for $(h-m)-$convex functions holds.

Theorem 8. *Let* $f : [0,\infty) \to \mathbb{R}$ *be a* $(h-m)-$*convex function with* $m \in (0,1]$*,* $t \in [0,1]$*. If* $0 \leq a < b < \infty$ *and* $f \in L_1[a,b]$*, then the following inequality holds;*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &\leq \min\left\{ f(a)\int_0^1 h(t)\,dt + mf\left(\frac{b}{m}\right)\int_0^1 h(1-t)\,dt, \right. \\ &\qquad \left. f(b)\int_0^1 h(t)\,dt + mf\left(\frac{a}{m}\right)\int_0^1 h(1-t)\,dt \right\}. \end{aligned} \tag{2.4}$$

Proof. From the definition of $(h-m)-$convex functions, we can write

$$f\left(tx+m\left(1-t\right)y\right)\leq h\left(t\right)f\left(x\right)+mh\left(1-t\right)f\left(y\right)$$

for all $x,y\geq 0$. It follows that; for all $t\in[0,1]$,

$$f\left(ta+\left(1-t\right)b\right)\leq h\left(t\right)f\left(a\right)+mh\left(1-t\right)f\left(\frac{b}{m}\right)$$

and

$$f\left(tb+\left(1-t\right)a\right)\leq h\left(t\right)f\left(b\right)+mh\left(1-t\right)f\left(\frac{a}{m}\right).$$

Integrating these inequalities on $[0,1]$, with respect to t, we obtain

$$\int_0^1 f\left(ta+\left(1-t\right)b\right)dt\leq f\left(a\right)\int_0^1 h\left(t\right)dt+mf\left(\frac{b}{m}\right)\int_0^1 h\left(1-t\right)dt$$

and

$$\int_0^1 f\left(tb+\left(1-t\right)a\right)dt\leq f\left(b\right)\int_0^1 h\left(t\right)dt+mf\left(\frac{a}{m}\right)\int_0^1 h\left(1-t\right)dt.$$

It is easy to see that;

$$\int_0^1 f\left(ta+\left(1-t\right)b\right)dt=\int_0^1 f\left(tb+\left(1-t\right)a\right)dt=\frac{1}{b-a}\int_a^b f(x)dx.$$

Using this equality, we obtain the required result. □

Corollary 1. *If we choose $h(t)=1$ in (2.4), we obtain the following inequality;*

$$\frac{1}{b-a}\int_a^b f(x)dx\leq \min\left\{f\left(a\right)+mf\left(\frac{b}{m}\right),\ f\left(b\right)+mf\left(\frac{a}{m}\right)\right\}.$$

Corollary 2. *If we choose $m=1$ in (2.4), we obtain the following inequality;*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)dx \quad\leq\quad &\min\left\{f\left(a\right)\int_0^1 h\left(t\right)dt+f\left(b\right)\int_0^1 h\left(1-t\right)dt,\right.\\ &\left.f\left(b\right)\int_0^1 h\left(t\right)dt+f\left(a\right)\int_0^1 h\left(1-t\right)dt\right\}.\end{aligned}$$

Remark 2. *If we choose $h(t)=t$ in (2.4), we obtain the inequality (1.1).*

Remark 3. *If we choose $m=1$ and $h(t)=t$ in (2.4), we obtain the right hand side of the Hadamard's inequality. If we choose $m=1$ and $h(t)=1$ in (2.4), we obtain the right hand side of the inequality (1.6). If we choose $m=1$ and $h(t)=t^s$ in (2.4), we obtain the right hand side of the inequality (1.4).*

Another result of Hadamard-type for $(h-m)-$convex functions is emboided in the following theorem.

Theorem 9. *Let $f:[0,\infty)\to\mathbb{R}$ be a $(h-m)-$convex function with $m\in(0,1]$, $t\in[0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[a,b]$, then the following inequality holds;*

$$
\begin{aligned}
(2.5)\qquad f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b\left[f(x)+mf\left(\frac{x}{m}\right)\right]dx \\
&\leq h\left(\frac{1}{2}\right)\left[\frac{f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right)}{2}\right]
\end{aligned}
$$

Proof. For $x,y\in[0,\infty)$ and $\alpha=\frac{1}{2}$, we can write definition of $(h-m)-$convex function as following;

$$f\left(\frac{x+y}{2}\right)\leq h\left(\frac{1}{2}\right)f(x)+mh\left(\frac{1}{2}\right)f\left(\frac{y}{m}\right)$$

If we choose $x=ta+(1-t)b$ and $y=tb+(1-t)a$, we get

$$f\left(\frac{a+b}{2}\right)\leq h\left(\frac{1}{2}\right)f(ta+(1-t)b)+mh\left(\frac{1}{2}\right)f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)$$

for all $t\in[0,1]$. By integrating the result on $[0,1]$ with respect to t, we have
(2.6)

$$f\left(\frac{a+b}{2}\right)\leq h\left(\frac{1}{2}\right)\int_0^1 f(ta+(1-t)b)\,dt+mh\left(\frac{1}{2}\right)\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt.$$

By the facts that

$$\int_0^1 f(ta+(1-t)b)\,dt=\frac{1}{b-a}\int_a^b f(x)dx$$

and

$$\int_0^1 f\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)dt=\frac{m}{b-a}\int_{\frac{a}{m}}^{\frac{b}{m}} f(x)dx=\frac{1}{b-a}\int_a^b f\left(\frac{x}{m}\right)dx.$$

Using these equalities in (2.6), we obtain the first inequality of (2.5). By the $(h-m)-$convexity of f, we can write

$$
\begin{aligned}
(2.7)\qquad & h\left(\frac{1}{2}\right)\left[f(ta+(1-t)b)+mf\left((1-t)\frac{a}{m}+t\frac{b}{m}\right)\right] \\
\leq\ & h\left(\frac{1}{2}\right)\left[tf(a)+m(1-t)f\left(\frac{b}{m}\right)+m(1-t)f\left(\frac{a}{m}\right)+m^2tf\left(\frac{b}{m^2}\right)\right].
\end{aligned}
$$

Integrating the inequality (2.7) on on $[0,1]$ with respect to t, we have

$$
\begin{aligned}
& \frac{h\left(\frac{1}{2}\right)}{b-a}\left[\int_a^b f(x)dx+m\int_a^b f\left(\frac{x}{m}\right)dx\right] \\
\leq\ & h\left(\frac{1}{2}\right)\left[\frac{f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2f\left(\frac{b}{m^2}\right)}{2}\right]
\end{aligned}
$$

which completes the proof. $\square$

Corollary 3. *If we choose $h(t)=1$ in (2.5), we obtain the following inequality;*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a}\int_a^b \left[f(x)+mf\left(\frac{x}{m}\right)\right]dx \\ &\leq \left[\frac{f(a)+mf\left(\frac{b}{m}\right)+mf\left(\frac{a}{m}\right)+m^2 f\left(\frac{b}{m^2}\right)}{2}\right]. \end{aligned}$$

Corollary 4. *If we choose $m=1$ and $h(t)=t^s$ in (2.5), we obtain the following inequality;*

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}$$

which is similar to (1.4).

Remark 4. *If we choose $m=1$ in (2.4), we obtain the right hand side of the inequality (1.8).*

Remark 5. *If we choose $m=1$ and $h(t)=t$ in (2.5), we obtain the Hadamard's inequality.*

Remark 6. *If we choose $h(t)=t$ in (2.5), we obtain the inequality (1.2).*

The following inequality also holds for $(h-m)-$convex functions.

Theorem 10. *Let $f:[0,\infty)\to\mathbb{R}$ be a $(h-m)-$convex function with $m\in(0,1]$, $t\in[0,1]$. If $0\leq a<b<\infty$ and $f\in L_1[ma,b]$, then the following inequality holds;*

$$\begin{aligned} &\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)dx+\frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right] \\ &\leq \frac{f(a)+f(b)}{2}\left[\int_0^1 h(t)\,dt+\int_0^1 h(1-t)\,dt\right]. \end{aligned} \tag{2.8}$$

Proof. From definition of $(h-m)-$convex functions, we can write

$$f(ta+m(1-t)b)\leq h(t)f(a)+mh(1-t)f(b)$$

$$f((1-t)a+mtb)\leq h(1-t)f(a)+mh(t)f(b)$$

$$f(tb+m(1-t)a)\leq h(t)f(b)+mh(1-t)f(a)$$

and

$$f((1-t)b+mta)\leq h(1-t)f(b)+mh(t)f(a)$$

for all $t \in [0,1]$. By summing these inequalities and integrating on $[0,1]$ with respect to t, we obtain

$$
\begin{aligned}
(2.9) \qquad & \int_0^1 f(ta+m(1-t)b)\,dt + \int_0^1 f((1-t)a+mtb)\,dt \\
& + \int_0^1 f(tb+m(1-t)a)\,dt + \int_0^1 f((1-t)b+mta)\,dt \\
\leq \quad & (f(a)+f(b))(m+1)\left[\int_0^1 h(t)\,dt + \int_0^1 h(1-t)\,dt\right].
\end{aligned}
$$

It is easy to show that

$$\int_0^1 f(ta+m(1-t)b)\,dt = \int_0^1 f((1-t)a+mtb)\,dt = \frac{1}{mb-a}\int_a^{mb} f(x)dx$$

and

$$\int_0^1 f(tb+m(1-t)a)\,dt = \int_0^1 f((1-t)b+mta)\,dt = \frac{1}{b-ma}\int_{ma}^{b} f(x)dx.$$

By using these equalities in (2.9), we get the desired result. □

Corollary 5. *If we choose $h(t)=1$ in (2.8), we obtain the following inequality;*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{ma}^{b} f(x)dx\right] \leq f(a)+f(b).$$

Remark 7. *If we choose $m=1$ and $h(t)=t$ in (2.8), we obtain the right hand side of the Hadamard's inequality. If we choose $m=1$ and $h(t)=1$ in (2.8), we obtain the right hand side of the inequality (1.6). If we choose $m=1$ and $h(t)=t^s$ in (2.8), we obtain the right hand side of the inequality (1.4).*

Remark 8. *If we choose $h(t)=t$ in (2.8), we obtain the inequality (1.3).*

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